# Gyrokinetic limit of the 2D Hartree equation in a large magnetic field


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## Model

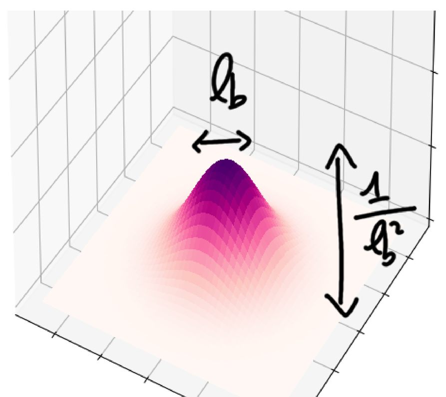
Motivated in particular by the physical context of the quantum Hall effect [5], we study a large system of spinless, non relativistic fermions in $\mathbb{R}^2$ subjected to a homogeneous transverse magnetic field of amplitude $b$.

The kinetic energy is given by the magnetic Laplacian in symmetric gauge:

$$\mathscr{L}_b \coloneqq \left(i\hbar\nabla + \frac{b}{2}X^\perp\right)^2$$

where $\hbar$ is our semi-classical parameter and $l_b \coloneqq \hbar/b$ the magnetic length. We consider a large magnetic field/semi-classical limit

$$l_b \to 0, \quad \hbar b = \mathcal{O}(1). \qquad \text{(S)}$$



Ground state density of $\mathscr{L}_b$.

Let $V$ be the external potential and $w$ the interaction potential, assumed to be radial. We study the solution $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ to Hartree's equation:

$$i l_b^2 \partial_t \gamma = \left[\mathscr{L}_b + V + w \star \rho_\gamma, \gamma\right], \qquad \text{(H)}$$

where $\mathcal{L}^1$ is the trace class and $\rho_\gamma(t,x) \coloneqq \gamma(t)(x,x)$ the density associated to $\gamma$ that we identify with its integral kernel.

We impose that the initial data satisfies

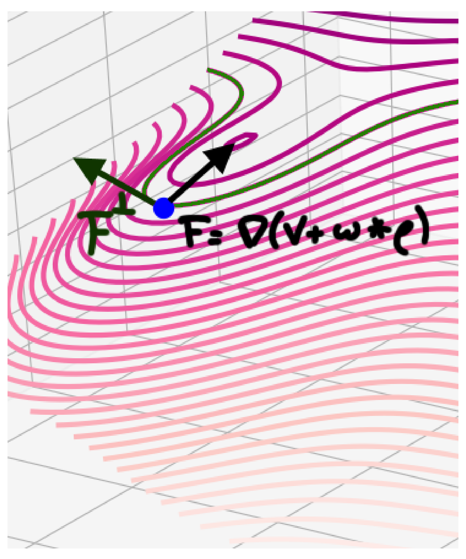
$$\operatorname{Tr}(\gamma(0)) = 1, \qquad 0 \leqslant \gamma(0) \leqslant 2\pi l_b^2. \qquad \text{(PP)}$$

With the scaling (S) all the terms in the Hamiltonian $\mathscr{L}_b + V + w \star \rho_\gamma$ are $\mathcal{O}(1)$.

Our goal is to obtain from (H) the drift equation

$$\partial_t \rho + \nabla^\perp (V + w \star \rho) \cdot \nabla \rho = 0 \qquad \text{(D)}$$

for a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$.



Level sets of $V + w \star \rho$.

**Remark:** *Pauli principle*

(PP) guarantees that the system occupies an $\mathcal{O}(1)$ volume. It is known that the degeneracy per area inside a Landau level is of order $l_b^{-2}$ [3]. A typical fermionic state satisfying (PP) is a projection onto a $N$-body Slater determinant of $N \coloneqq \mathcal{O}(l_b^{-2})$ orthonormal one body wave-functions.

## Classical trajectories

The limiting dynamics can be guessed by studying that of a classical particle of charge $-1$ submited to a force field $F$. Newton's fundamental equation of dynamics gives

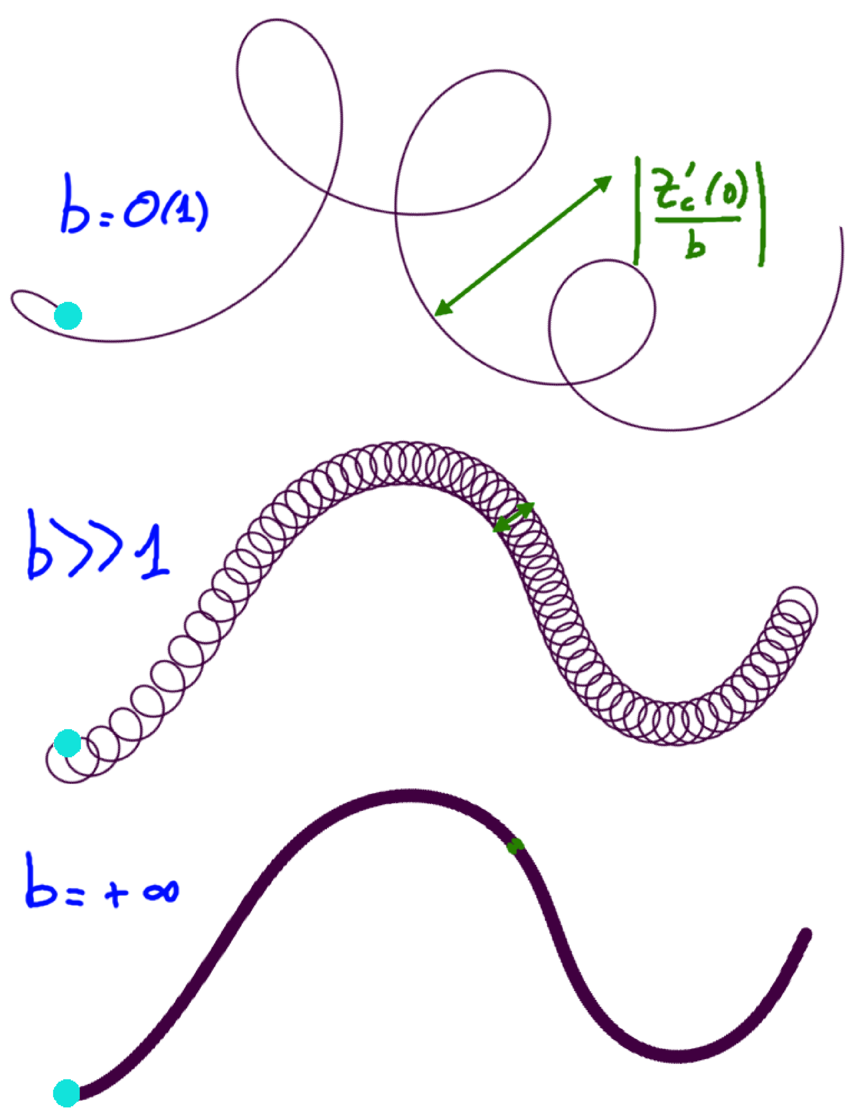
$$Z''(t) = F(t, Z(t)) + bZ'(t)^\perp.$$

For a constant and homogeneous force field, the motion splits into a cyclotron orbit and a drift of the orbit's center:

$$Z(t) = \underbrace{\frac{|Z_c'(0)|}{b}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix}}_{=:Z_c(t)} + \underbrace{\frac{F^\perp}{b}t}_{=:Z_d(t)}$$

where we assumed

$$Z_d(0) = (0,0),$$
$$Z_c(0) = \frac{|Z_c'(0)|}{b}(1,0).$$



Classical trajectories for different $b$.

The characteristic time for the cyclotron orbit is $b^{-1}$ while that for the drift is of order $b$. This suggests, for $b \to \infty$, to observe the motion over a time scale of order $b$. Noticing that the usual semi-classical time scale is $\hbar^{-1}$ [1], this explain the large magnetic field/semi-classical time scale $l_b^{-2} = b/\hbar$ in (H). The fast cyclotron motion will be averaged over in the limit equation, its' radius being given by $|Z_c'(0)|/b$.

Then, if we assume a more general force field $F$, slowly varying on the scale of the cyclotron orbit, we should expect to leading order an effective equation $Z_d'(t) = F^\perp/b$ for the motion of the orbit center. Following (H) we should set $F \coloneqq \nabla(V + w \star \rho)$ which leads via the method of characteristics to (D) which is a transport equation with force field $F$.

## Main result

We denote $\Gamma(\mu,\nu)$ the set of couplings between probabilities $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^2\right)$ and $W_1(\mu,\nu) \coloneqq \inf_{\pi \in \Gamma(\mu,\nu)} \iint |x-y|\, d\pi(x,y)$ the Wasserstein metric.

**Theorem:** *Convergence of densities*

Let $\gamma$ be the solution of (H) with initial data satisfying (PP) and for some $p > 7$,

$$\operatorname{Tr}\left(\gamma(0)\left(\mathscr{L}_b + V + \frac{1}{2}w * \rho_\gamma\right)\right) \leqslant C, \quad \operatorname{Tr}\left(\gamma(0)|X|^p\right) \leqslant C$$

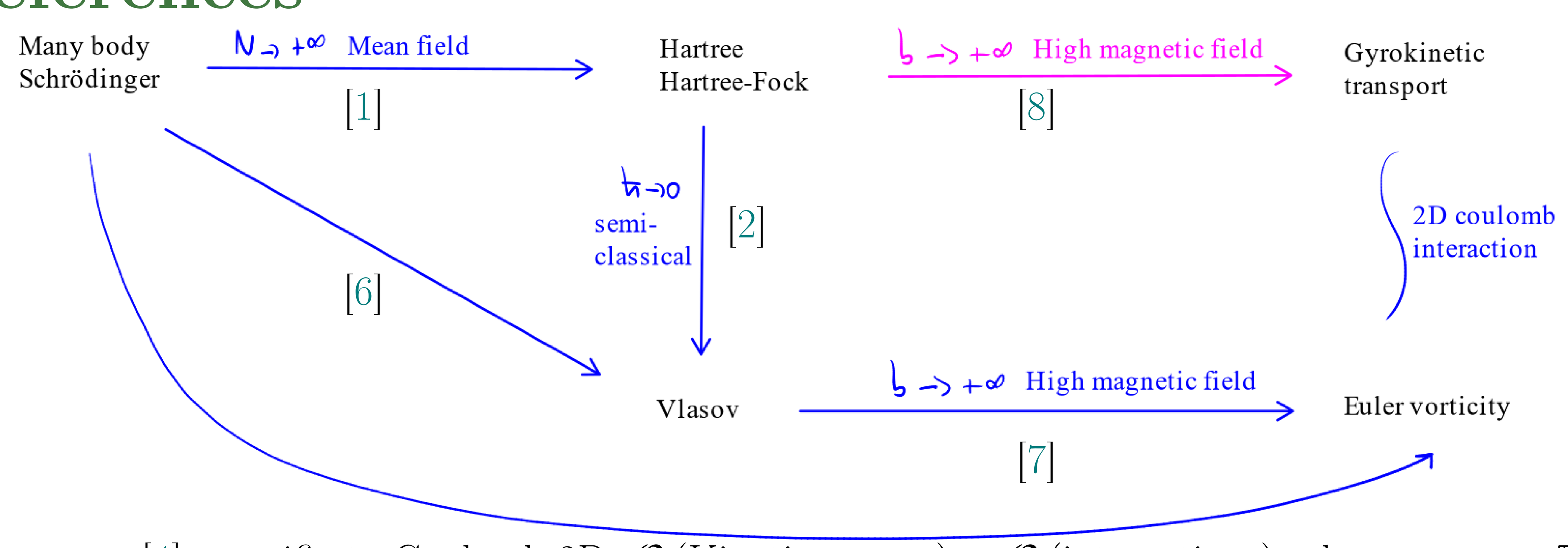
Let $\rho$ solve (D). Assume $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$ and $\nabla w \in L^1\left(\mathbb{R}^2\right), w \in H^2\left(\mathbb{R}^2\right)$. Then $\forall t \in \mathbb{R}_+, \forall \varphi \in W^{1,\infty}\left(\mathbb{R}^2\right) \cap H^2\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_\gamma(t) - \rho(t)\right)\right| \leqslant \widetilde{C}(C,p,t,V,w)\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(W_1\left(\rho_\gamma(0),\rho(0)\right) + l_b^{\min\left(2\frac{p-7}{4p-7},\frac{2}{7}\right)}\right)$$

For the proof of this result, we use vortex coherent states and the associated Husimi function to define a semi-classical density almost satisfying the limiting equation. We then deduce convergence of the density of the Hartree solution by a Dobrushin-type stability estimate for the limiting equation.

Main difficulties arise from
- the Landau level quantization of the momentum in the semi-classical phase space $\mathbb{R}^2 \times \mathbb{N}$ due to the spectral gap $\hbar b = \mathcal{O}(1)$,
- the fast cyclotron motion creating oscillations that have to be smoothed out against regular test functions,
- the large magnetic time scale, making propagating a semi-classical structure challenging.

## References




[4]: specific to Coulomb 2D, $\mathcal{O}$ (Kinetic energy) $\ll \mathcal{O}$ (interactions), phase space $= \mathbb{R}^2 \times \mathbb{R}^2$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